CHAPTER 13

MAGNETICALLY COUPLED CIRCUITS

People want success but keep running away from problems, and yet it is only in tackling problems that success is achieved.

— Josiah J. Bonire

Enhancing Your Career

Career in Electromagnetics Electromagnetics is the branch of electrical engineering (or physics) that deals with the analysis and application of electric and magnetic fields. In electromagnetics, electric circuit analysis is applied at low frequencies.

The principles of electromagnetics (EM) are applied in various allied disciplines, such as electric machines, electromechanical energy conversion, radar meteorology, remote sensing, satellite communications, bioelectromagnetics, electromagnetic interference and compatibility, plasmas, and fiber optics. EM devices include electric motors and generators, transformers, electromagnets, magnetic levitation, antennas, radars, microwave ovens, microwave dishes, superconductors, and electrocardiograms. The design of these devices requires a thorough knowledge of the laws and principles of EM.

EM is regarded as one of the more difficult disciplines in electrical engineering. One reason is that EM phenomena are rather abstract. But if one enjoys working with mathematics and can visualize the invisible, one should consider being a specialist in EM, since few electrical engineers specialize in this area. Electrical engineers who specialize in EM are needed in microwave industries, radio/TV broadcasting stations, electromagnetic research laboratories, and several communications industries.

Telemetry receiving station for space satellites. Source: T. J. Maloney, Modern Industrial Electronics, 3rd ed. Englewood Cliffs, NJ: Prentice Hall, 1996, p. 718.

13.1 INTRODUCTION

The circuits we have considered so far may be regarded as *conductively coupled*, because one loop affects the neighboring loop through current conduction. When two loops with or without contacts between them affect each other through the magnetic field generated by one of them, they are said to be *magnetically coupled*.

The transformer is an electrical device designed on the basis of the concept of magnetic coupling. It uses magnetically coupled coils to transfer energy from one circuit to another. Transformers are key circuit elements. They are used in power systems for stepping up or stepping down ac voltages or currents. They are used in electronic circuits such as radio and television receivers for such purposes as impedance matching, isolating one part of a circuit from another, and again for stepping up or down ac voltages and currents.

We will begin with the concept of mutual inductance and introduce the dot convention used for determining the voltage polarities of inductively coupled components. Based on the notion of mutual inductance, we then introduce the circuit element known as the *transformer*. We will consider the linear transformer, the ideal transformer, the ideal autotransformer, and the three-phase transformer. Finally, among their important applications, we look at transformers as isolating and matching devices and their use in power distribution.

13.2 MUTUAL INDUCTANCE

When two inductors (or coils) are in a close proximity to each other, the magnetic flux caused by current in one coil links with the other coil, thereby inducing voltage in the latter. This phenomenon is known as *mutual inductance*.

Let us first consider a single inductor, a coil with N turns. When current i flows through the coil, a magnetic flux ϕ is produced around it (Fig. 13.1). According to Faraday's law, the voltage v induced in the coil is proportional to the number of turns N and the time rate of change of the magnetic flux ϕ; that is,

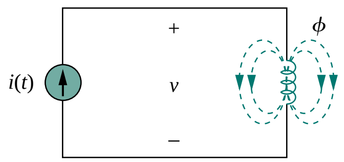


Figure 13.1 Magnetic flux produced by a single coil with N turns.

$$v = N\frac{d\phi}{dt} \tag{13.1}$$

But the flux ϕ is produced by current i so that any change in ϕ is caused by a change in the current. Hence, Eq. (13.1) can be written as

$$v = N\frac{d\phi}{di}\frac{di}{dt} \tag{13.2}$$

or

$$v = L\frac{di}{dt} \tag{13.3}$$

which is the voltage-current relationship for the inductor. From Eqs. (13.2) and (13.3), the inductance L of the inductor is thus given by

$$L = N\frac{d\phi}{di} \tag{13.4}$$

This inductance is commonly called *self-inductance*, because it relates the voltage induced in a coil by a time-varying current in the same coil.

Now consider two coils with self-inductances L_1 and L_2 that are in close proximity with each other (Fig. 13.2). Coil 1 has N_1 turns, while coil 2 has N_2 turns. For the sake of simplicity, assume that the second inductor carries no current. The magnetic flux ϕ_1 emanating from coil 1 has two components: one component ϕ_{11} links only coil 1, and another component ϕ_{12} links both coils. Hence,

$$\phi_1 = \phi_{11} + \phi_{12} \tag{13.5}$$

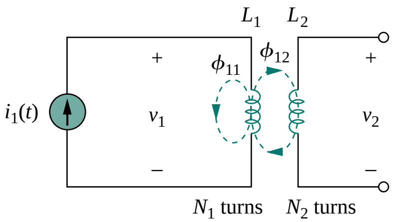


Figure 13.2 Mutual inductance M_{21} of coil 2 with respect to coil 1.

Although the two coils are physically separated, they are said to be *magnetically coupled*. Since the entire flux ϕ_1 links coil 1, the voltage induced in coil 1 is

$$v_1 = N_1 \frac{d\phi_1}{dt} \tag{13.6}$$

Only flux ϕ_{12} links coil 2, so the voltage induced in coil 2 is

$$v_2 = N_2 \frac{d\phi_{12}}{dt} \tag{13.7}$$

Again, as the fluxes are caused by the current i_1 flowing in coil 1, Eq. (13.6) can be written as

$$v_1 = N_1 \frac{d\phi_1}{di_1}\frac{di_1}{dt} = L_1 \frac{di_1}{dt} \tag{13.8}$$

where $L_1 = N_1\, d\phi_1/di_1$ is the self-inductance of coil 1. Similarly, Eq. (13.7) can be written as

$$v_2 = N_2 \frac{d\phi_{12}}{di_1}\frac{di_1}{dt} = M_{21} \frac{di_1}{dt} \tag{13.9}$$

where

$$M_{21} = N_2 \frac{d\phi_{12}}{di_1} \tag{13.10}$$

M_{21} is known as the *mutual inductance* of coil 2 with respect to coil 1. Subscript 21 indicates that the inductance M_{21} relates the voltage induced in coil 2 to the current in coil 1. Thus, the open-circuit *mutual voltage* (or induced voltage) across coil 2 is

$$\boxed{v_2 = M_{21} \frac{di_1}{dt}} \tag{13.11}$$

Suppose we now let current i_2 flow in coil 2, while coil 1 carries no current (Fig. 13.3). The magnetic flux ϕ_2 emanating from coil 2 comprises flux ϕ_{22} that links only coil 2 and flux ϕ_{21} that links both coils. Hence,

$$\phi_2 = \phi_{21} + \phi_{22} \tag{13.12}$$

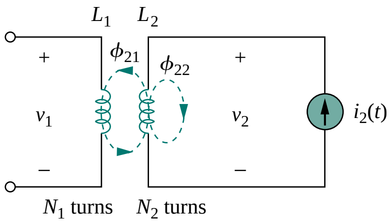


Figure 13.3 Mutual inductance M_{12} of coil 1 with respect to coil 2.

The entire flux ϕ_2 links coil 2, so the voltage induced in coil 2 is

$$v_2 = N_2 \frac{d\phi_2}{dt} = N_2 \frac{d\phi_2}{di_2}\frac{di_2}{dt} = L_2 \frac{di_2}{dt} \tag{13.13}$$

where $L_2 = N_2\, d\phi_2/di_2$ is the self-inductance of coil 2. Since only flux ϕ_{21} links coil 1, the voltage induced in coil 1 is

$$v_1 = N_1 \frac{d\phi_{21}}{dt} = N_1 \frac{d\phi_{21}}{di_2}\frac{di_2}{dt} = M_{12}\frac{di_2}{dt} \tag{13.14}$$

where

$$M_{12} = N_1 \frac{d\phi_{21}}{di_2} \tag{13.15}$$

which is the *mutual inductance* of coil 1 with respect to coil 2. Thus, the open-circuit *mutual voltage* across coil 1 is

$$\boxed{v_1 = M_{12}\frac{di_2}{dt}} \tag{13.16}$$

We will see in the next section that M_{12} and M_{21} are equal, that is,

$$M_{12} = M_{21} = M \tag{13.17}$$

and we refer to M as the mutual inductance between the two coils. Like self-inductance L, mutual inductance M is measured in henrys (H). Keep in mind that mutual coupling only exists when the inductors or coils are in close proximity, and the circuits are driven by time-varying sources. We recall that inductors act like short circuits to dc.

From the two cases in Figs. 13.2 and 13.3, we conclude that mutual inductance results if a voltage is induced by a time-varying current in another circuit. It is the property of an inductor to produce a voltage in reaction to a time-varying current in another inductor near it. Thus,

> **Mutual inductance** is the ability of one inductor to induce a voltage across a neighboring inductor, measured in henrys (H).

Although mutual inductance M is always a positive quantity, the mutual voltage $M\, di/dt$ may be negative or positive, just like the self-induced voltage $L\, di/dt$. However, unlike the self-induced $L\, di/dt$, whose polarity is determined by the reference direction of the current and the reference polarity of the voltage (according to the passive sign convention), the polarity of mutual voltage $M\, di/dt$ is not easy to determine, because four terminals are involved. The choice of the correct polarity for $M\, di/dt$ is made by examining the orientation or particular way in which both coils are physically wound and applying Lenz's law in conjunction with the right-hand rule. Since it is inconvenient to show the construction details of coils on a circuit schematic, we apply the *dot convention* in circuit analysis. By this convention, a dot is placed in the circuit at one end of each of the two magnetically coupled coils to indicate the direction of the magnetic flux if current enters that dotted terminal of the coil. This is illustrated in Fig. 13.4. Given a circuit, the dots are already placed beside the coils so that we need not bother about how to place them. The dots are used along with the dot convention to determine the polarity of the mutual voltage. The dot convention is stated as follows:

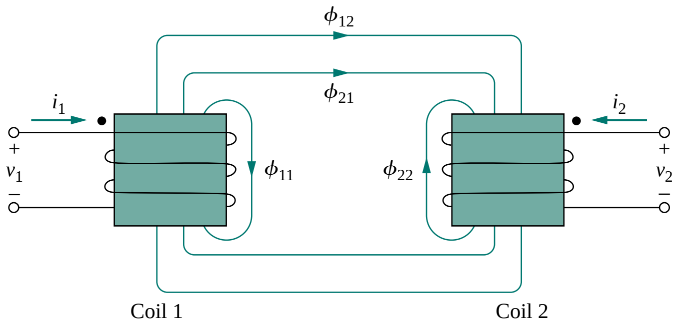


Figure 13.4 Illustration of the dot convention.

> If a current enters the dotted terminal of one coil, the reference polarity of the mutual voltage in the second coil is positive at the dotted terminal of the second coil.

Alternatively,

> If a current leaves the dotted terminal of one coil, the reference polarity of the mutual voltage in the second coil is negative at the dotted terminal of the second coil.

Thus, the reference polarity of the mutual voltage depends on the reference direction of the inducing current and the dots on the coupled coils. Application of the dot convention is illustrated in the four pairs of mutually coupled coils in Fig. 13.5. For the coupled coils in Fig. 13.5(a), the sign of the mutual voltage v_2 is determined by the reference polarity for v_2 and the direction of i_1. Since i_1 enters the dotted terminal of coil 1 and v_2 is positive at the dotted terminal of coil 2, the mutual voltage is $+M\,di_1/dt$. For the coils in Fig. 13.5(b), the current i_1 enters the dotted terminal of coil 1 and v_2 is negative at the dotted terminal of coil 2. Hence, the mutual voltage is $-M\,di_1/dt$. The same reasoning applies to the coils in Fig. 13.5(c) and 13.5(d). Figure 13.6 shows the dot convention for coupled coils in series. For the coils in Fig. 13.6(a), the total inductance is

$$L = L_1 + L_2 + 2M \qquad \text{(Series-aiding connection)} \tag{13.18}$$

For the coil in Fig. 13.6(b),

$$L = L_1 + L_2 - 2M \qquad \text{(Series-opposing connection)} \tag{13.19}$$

Now that we know how to determine the polarity of the mutual voltage, we are prepared to analyze circuits involving mutual inductance.

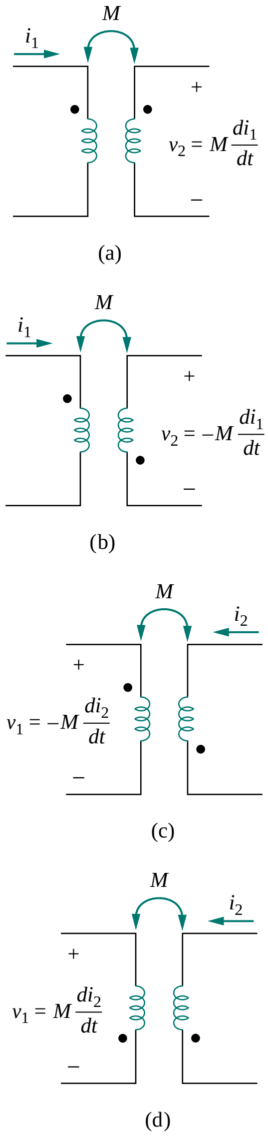


Figure 13.5 Examples illustrating how to apply the dot convention.

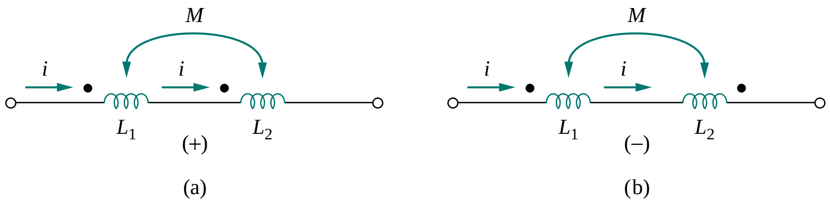


Figure 13.6 Dot convention for coils in series; the sign indicates the polarity of the mutual voltage: (a) series-aiding connection, (b) series-opposing connection.

As the first example, consider the circuit in Fig. 13.7. Applying KVL to coil 1 gives

$$v_1 = i_1 R_1 + L_1 \frac{di_1}{dt} + M \frac{di_2}{dt} \tag{13.20a}$$

For coil 2, KVL gives

$$v_2 = i_2 R_2 + L_2 \frac{di_2}{dt} + M \frac{di_1}{dt} \tag{13.20b}$$

We can write Eq. (13.20) in the frequency domain as

$$\mathbf{V}_1 = (R_1 + j\omega L_1)\mathbf{I}_1 + j\omega M \mathbf{I}_2 \tag{13.21a}$$

$$\mathbf{V}_2 = j\omega M \mathbf{I}_1 + (R_2 + j\omega L_2)\mathbf{I}_2 \tag{13.21b}$$

As a second example, consider the circuit in Fig. 13.8. We analyze this in the frequency domain. Applying KVL to coil 1, we get

$$\mathbf{V} = (\mathbf{Z}_1 + j\omega L_1)\mathbf{I}_1 - j\omega M \mathbf{I}_2 \tag{13.22a}$$

For coil 2, KVL yields

$$0 = -j\omega M \mathbf{I}_1 + (\mathbf{Z}_L + j\omega L_2)\mathbf{I}_2 \tag{13.22b}$$

Equations (13.21) and (13.22) are solved in the usual manner to determine the currents.

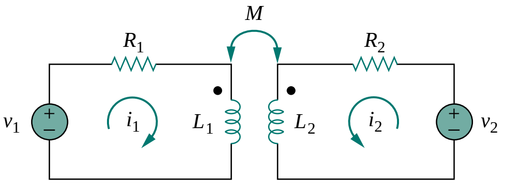


Figure 13.7 Time-domain analysis of a circuit containing coupled coils.

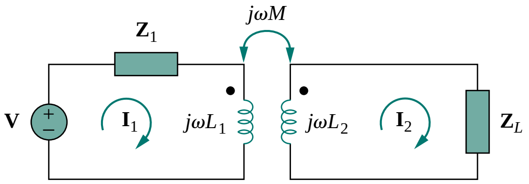
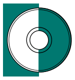


Figure 13.8 Frequency-domain analysis of a circuit containing coupled coils.

At this introductory level we are not concerned with the determination of the mutual inductances of the coils and their dot placements. Like R, L, and C, calculation of M would involve applying the theory of electromagnetics to the actual physical properties of the coils. In this text, we assume that the mutual inductance and the dots placement are the "givens" of the circuit problem, like the circuit components R, L, and C.

EXAMPLE 13.1

Calculate the phasor currents $\mathbf{I}_1$ and $\mathbf{I}_2$ in the circuit of Fig. 13.9.



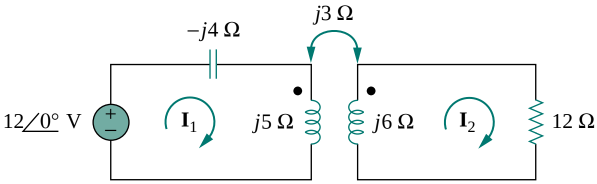


Figure 13.9 For Example 13.1.

Solution:

For coil 1, KVL gives

$$-12 + (-j4 + j5)\mathbf{I}_1 - j3\mathbf{I}_2 = 0$$

or

$$j\mathbf{I}_1 - j3\mathbf{I}_2 = 12 \tag{13.1.1}$$

For coil 2, KVL gives

$$-j3\mathbf{I}_1 + (12 + j6)\mathbf{I}_2 = 0$$

or

$$\mathbf{I}_1 = \frac{(12 + j6)\mathbf{I}_2}{j3} = (2 - j4)\mathbf{I}_2 \tag{13.1.2}$$

Substituting this in Eq. (13.1.1), we get

$$(j2 + 4 - j3)\mathbf{I}_2 = (4 - j)\mathbf{I}_2 = 12$$

or

$$\mathbf{I}_2 = \frac{12}{4 - j} = 2.91\angle 14.04° \text{ A} \tag{13.1.3}$$

From Eqs. (13.1.2) and (13.1.3),

$$\mathbf{I}_1 = (2 - j4)\mathbf{I}_2 = (4.472\angle -63.43°)(2.91\angle 14.04°)$$
$$= 13.01\angle -49.39° \text{ A}$$

PRACTICE PROBLEM 13.1

Determine the voltage $\mathbf{V}_o$ in the circuit of Fig. 13.10.

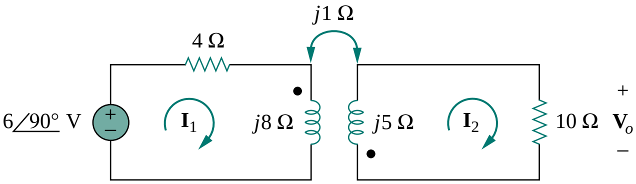


Figure 13.10 For Practice Prob. 13.1.

Answer: $0.6\angle -90°$ V.

EXAMPLE 13.2

Calculate the mesh currents in the circuit of Fig. 13.11.

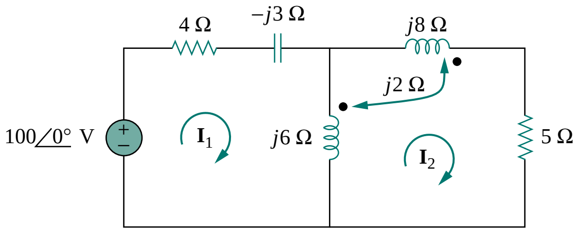


Figure 13.11 For Example 13.2.

Solution:

The key to analyzing a magnetically coupled circuit is knowing the polarity of the mutual voltage. We need to apply the dot rule. In Fig. 13.11, suppose coil 1 is the one whose reactance is 6 Ω, and coil 2 is the one whose reactance is 8 Ω. To figure out the polarity of the mutual voltage in coil 1 due to current $\mathbf{I}_2$, we observe that $\mathbf{I}_2$ leaves the dotted terminal of coil 2. Since we are applying KVL in the clockwise direction, it implies that the mutual voltage is negative, that is, $-j2\mathbf{I}_2$.

Alternatively, it might be best to figure out the mutual voltage by redrawing the relevant portion of the circuit, as shown in Fig. 13.12(a), where it becomes clear that the mutual voltage is $\mathbf{V}_1 = -2j\mathbf{I}_2$.

Thus, for mesh 1 in Fig. 13.11, KVL gives

$$-100 + \mathbf{I}_1(4 - j3 + j6) - j6\mathbf{I}_2 - j2\mathbf{I}_2 = 0$$

or

$$100 = (4 + j3)\mathbf{I}_1 - j8\mathbf{I}_2 \tag{13.2.1}$$

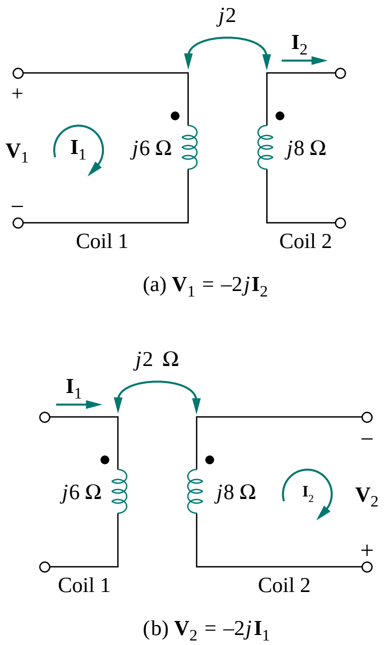


Figure 13.12 For Example 13.2; redrawing the relevant portion of the circuit in Fig. 13.11 to find mutual voltages by the dot convention.

Similarly, to figure out the mutual voltage in coil 2 due to current $\mathbf{I}_1$, consider the relevant portion of the circuit, as shown in Fig. 13.12(b). Applying the dot convention gives the mutual voltage as $\mathbf{V}_2 = -2j\mathbf{I}_1$. Also, current $\mathbf{I}_2$ sees the two coupled coils in series in Fig. 13.11; since it leaves the dotted terminals in both coils, Eq. (13.18) applies. Therefore, for mesh 2, KVL gives

$$0 = -2j\mathbf{I}_1 - j6\mathbf{I}_1 + (j6 + j8 + j2 \times 2 + 5)\mathbf{I}_2$$

or

$$0 = -j8\mathbf{I}_1 + (5 + j18)\mathbf{I}_2 \tag{13.2.2}$$

Putting Eqs. (13.2.1) and (13.2.2) in matrix form, we get

$$\begin{bmatrix} 100 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 + j3 & -j8 \\ -j8 & 5 + j18 \end{bmatrix} \begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \end{bmatrix}$$

The determinants are

$$\Delta = \begin{vmatrix} 4 + j3 & -j8 \\ -j8 & 5 + j18 \end{vmatrix} = 30 + j87$$

$$\Delta_1 = \begin{vmatrix} 100 & -j8 \\ 0 & 5 + j18 \end{vmatrix} = 100(5 + j18)$$

$$\Delta_2 = \begin{vmatrix} 4 + j3 & 100 \\ -j8 & 0 \end{vmatrix} = j800$$

Thus, we obtain the mesh currents as

$$\mathbf{I}_1 = \frac{\Delta_1}{\Delta} = \frac{100(5 + j18)}{30 + j87} = \frac{1868.2\angle 74.5^\circ}{92.03\angle 71^\circ} = 20.3\angle 3.5^\circ \text{ A}$$

$$\mathbf{I}_2 = \frac{\Delta_2}{\Delta} = \frac{j800}{30 + j87} = \frac{800\angle 90^\circ}{92.03\angle 71^\circ} = 8.693\angle 19^\circ \text{ A}$$

PRACTICE PROBLEM 13.2

Determine the phasor currents $\mathbf{I}_1$ and $\mathbf{I}_2$ in the circuit of Fig. 13.13.

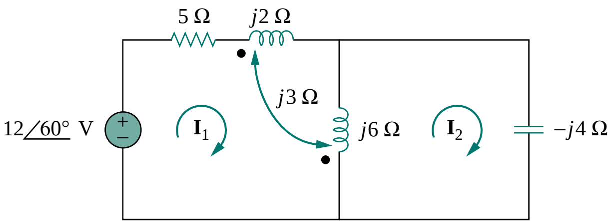


Figure 13.13 For Practice Prob. 13.2.

Answer: $2.15\angle 86.56^\circ$, $3.23\angle 86.56^\circ$ A.

13.3 ENERGY IN A COUPLED CIRCUIT

In Chapter 6, we saw that the energy stored in an inductor is given by

$$w = \frac{1}{2}Li^2 \tag{13.23}$$

We now want to determine the energy stored in magnetically coupled coils.

Consider the circuit in Fig. 13.14. We assume that currents i_1 and i_2 are zero initially, so that the energy stored in the coils is zero. If we let i_1 increase from zero to I_1 while maintaining $i_2 = 0$, the power in coil 1 is

$$p_1(t) = v_1 i_1 = i_1 L_1 \frac{di_1}{dt} \tag{13.24}$$

and the energy stored in the circuit is

$$w_1 = \int p_1\, dt = L_1 \int_0^{I_1} i_1\, di_1 = \frac{1}{2}L_1 I_1^2 \tag{13.25}$$

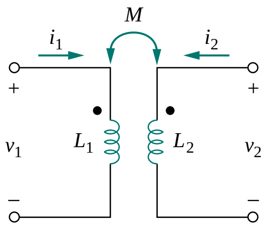


Figure 13.14 The circuit for deriving energy stored in a coupled circuit.

If we now maintain $i_1 = I_1$ and increase i_2 from zero to I_2, the mutual voltage induced in coil 1 is $M_{12}\, di_2/dt$, while the mutual voltage induced in coil 2 is zero, since i_1 does not change. The power in the coils is now

$$p_2(t) = i_1 M_{12}\frac{di_2}{dt} + i_2 v_2 = I_1 M_{12}\frac{di_2}{dt} + i_2 L_2 \frac{di_2}{dt} \tag{13.26}$$

and the energy stored in the circuit is

$$\begin{aligned} w_2 = \int p_2\, dt &= M_{12} I_1 \int_0^{I_2} di_2 + L_2 \int_0^{I_2} i_2\, di_2 \\ &= M_{12} I_1 I_2 + \frac{1}{2} L_2 I_2^2 \end{aligned} \tag{13.27}$$

The total energy stored in the coils when both i_1 and i_2 have reached constant values is

$$w = w_1 + w_2 = \frac{1}{2} L_1 I_1^2 + \frac{1}{2} L_2 I_2^2 + M_{12} I_1 I_2 \tag{13.28}$$

If we reverse the order by which the currents reach their final values, that is, if we first increase i_2 from zero to I_2 and later increase i_1 from zero to I_1, the total energy stored in the coils is

$$w = \frac{1}{2} L_1 I_1^2 + \frac{1}{2} L_2 I_2^2 + M_{21} I_1 I_2 \tag{13.29}$$

Since the total energy stored should be the same regardless of how we reach the final conditions, comparing Eqs. (13.28) and (13.29) leads us to conclude that

$$M_{12} = M_{21} = M \tag{13.30a}$$

and

$$w = \frac{1}{2} L_1 I_1^2 + \frac{1}{2} L_2 I_2^2 + M I_1 I_2 \tag{13.30b}$$

This equation was derived based on the assumption that the coil currents both entered the dotted terminals. If one current enters one dotted terminal while the other current leaves the other dotted terminal, the mutual voltage is negative, so that the mutual energy $M I_1 I_2$ is also negative. In that case,

$$w = \frac{1}{2} L_1 I_1^2 + \frac{1}{2} L_2 I_2^2 - M I_1 I_2 \tag{13.31}$$

Also, since I_1 and I_2 are arbitrary values, they may be replaced by i_1 and i_2, which gives the instantaneous energy stored in the circuit the general expression

$$\boxed{w = \frac{1}{2} L_1 i_1^2 + \frac{1}{2} L_2 i_2^2 \pm M i_1 i_2} \tag{13.32}$$

The positive sign is selected for the mutual term if both currents enter or leave the dotted terminals of the coils; the negative sign is selected otherwise.

We will now establish an upper limit for the mutual inductance M. The energy stored in the circuit cannot be negative because the circuit is passive. This means that the quantity $1/2L_1i_1^2 + 1/2L_2i_2^2 - Mi_1i_2$ must be greater than or equal to zero,

$$\frac{1}{2}L_1i_1^2 + \frac{1}{2}L_2i_2^2 - Mi_1i_2 \geq 0 \tag{13.33}$$

To complete the square, we both add and subtract the term $i_1i_2\sqrt{L_1L_2}$ on the right-hand side of Eq. (13.33) and obtain

$$\frac{1}{2}(i_1\sqrt{L_1} - i_2\sqrt{L_2})^2 + i_1i_2(\sqrt{L_1L_2} - M) \geq 0 \tag{13.34}$$

The squared term is never negative; at its least it is zero. Therefore, the second term on the right-hand side of Eq. (13.34) must be greater than zero; that is,

$$\sqrt{L_1L_2} - M \geq 0$$

or

$$M \leq \sqrt{L_1L_2} \tag{13.35}$$

Thus, the mutual inductance cannot be greater than the geometric mean of the self-inductances of the coils. The extent to which the mutual inductance M approaches the upper limit is specified by the *coefficient of coupling* k, given by

$$k = \frac{M}{\sqrt{L_1L_2}} \tag{13.36}$$

or

$$\boxed{M = k\sqrt{L_1L_2}} \tag{13.37}$$

where $0 \leq k \leq 1$ or equivalently $0 \leq M \leq \sqrt{L_1L_2}$. The coupling coefficient is the fraction of the total flux emanating from one coil that links the other coil. For example, in Fig. 13.2,

$$k = \frac{\phi_{12}}{\phi_1} = \frac{\phi_{12}}{\phi_{11} + \phi_{12}} \tag{13.38}$$

and in Fig. 13.3,

$$k = \frac{\phi_{21}}{\phi_2} = \frac{\phi_{21}}{\phi_{21} + \phi_{22}} \tag{13.39}$$

If the entire flux produced by one coil links another coil, then $k = 1$ and we have 100 percent coupling, or the coils are said to be *perfectly coupled*. Thus,

> The coupling coefficient k is a measure of the magnetic coupling between two coils; $0 \leq k \leq 1$.

For $k < 0.5$, coils are said to be *loosely coupled*; and for $k > 0.5$, they are said to be *tightly coupled*.

We expect k to depend on the closeness of the two coils, their core, their orientation, and their windings. Figure 13.15 shows loosely coupled

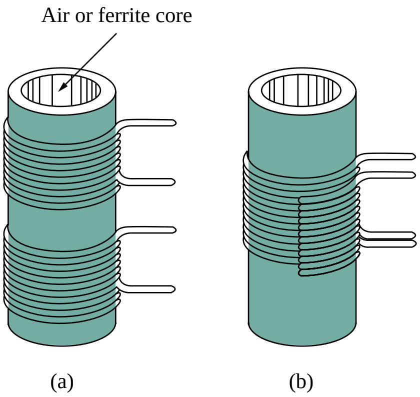


Figure 13.15 Windings: (a) loosely coupled, (b) tightly coupled; cutaway view demonstrates both windings.

windings and tightly coupled windings. The air-core transformers used in radio frequency circuits are loosely coupled, whereas iron-core transformers used in power systems are tightly coupled. The linear transformers discussed in Section 3.4 are mostly air-core; the ideal transformers discussed in Sections 13.5 and 13.6 are principally iron-core.

EXAMPLE 13.3

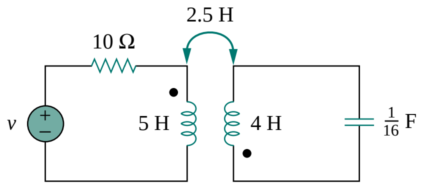


Figure 13.16 For Example 13.3.

Consider the circuit in Fig. 13.16. Determine the coupling coefficient. Calculate the energy stored in the coupled inductors at time $t = 1$ s if $v = 60\cos(4t + 30^\circ)$ V.

Solution:

The coupling coefficient is

$$k = \frac{M}{\sqrt{L_1 L_2}} = \frac{2.5}{\sqrt{20}} = 0.56$$

indicating that the inductors are tightly coupled. To find the energy stored, we need to obtain the frequency-domain equivalent of the circuit.

$$\begin{aligned} 60\cos(4t + 30^\circ) &\implies 60\underline{/30^\circ}, \quad \omega = 4 \text{ rad/s} \\ 5\text{ H} &\implies j\omega L_1 = j20\ \Omega \\ 2.5\text{ H} &\implies j\omega M = j10\ \Omega \\ 4\text{ H} &\implies j\omega L_2 = j16\ \Omega \\ \frac{1}{16}\text{ F} &\implies \frac{1}{j\omega C} = -j4\ \Omega \end{aligned}$$

The frequency-domain equivalent is shown in Fig. 13.17. We now apply mesh analysis. For mesh 1,

$$(10 + j20)\mathbf{I}_1 + j10\mathbf{I}_2 = 60\underline{/30^\circ} \tag{13.3.1}$$

For mesh 2,

$$j10\mathbf{I}_1 + (j16 - j4)\mathbf{I}_2 = 0$$

or

$$\mathbf{I}_1 = -1.2\mathbf{I}_2 \tag{13.3.2}$$

Substituting this into Eq. (13.3.1) yields

$$\mathbf{I}_2(-12 - j14) = 60\underline{/30^\circ} \quad \implies \quad \mathbf{I}_2 = 3.254\underline{/-160.6^\circ} \text{ A}$$

and

$$\mathbf{I}_1 = -1.2\mathbf{I}_2 = 3.905\underline{/-19.4^\circ} \text{ A}$$

In the time-domain,

$$i_1 = 3.905\cos(4t - 19.4^\circ), \qquad i_2 = 3.254\cos(4t - 199.4^\circ)$$

At time $t = 1$ s, $4t = 4$ rad $= 229.2^\circ$, and

$$i_1 = 3.905\cos(229.2^\circ - 19.4^\circ) = -3.389 \text{ A}$$
$$i_2 = 3.254\cos(229.2^\circ + 160.6^\circ) = 2.824 \text{ A}$$

The total energy stored in the coupled inductors is

$$w = \frac{1}{2}L_1 i_1^2 + \frac{1}{2}L_2 i_2^2 + M i_1 i_2$$

$$= \frac{1}{2}(5)(-3.389)^2 + \frac{1}{2}(4)(2.824)^2 + 2.5(-3.389)(2.824) = 20.73 \text{ J}$$

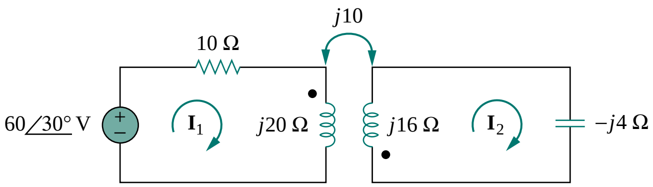


Figure 13.17 Frequency-domain equivalent of the circuit in Fig. 13.16.

PRACTICE PROBLEM 13.3

For the circuit in Fig. 13.18, determine the coupling coefficient and the energy stored in the coupled inductors at $t = 1.5$ s.

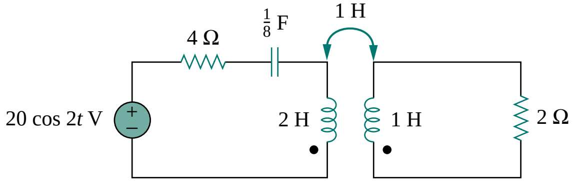


Figure 13.18 For Practice Prob. 13.3.

Answer: 0.7071, 9.85 J.

13.4 LINEAR TRANSFORMERS

Here we introduce the transformer as a new circuit element. A transformer is a magnetic device that takes advantage of the phenomenon of mutual inductance.

> A **transformer** is generally a four-terminal device comprising two (or more) magnetically coupled coils.

As shown in Fig. 13.19, the coil that is directly connected to the voltage source is called the *primary winding*. The coil connected to the load is called the *secondary winding*. The resistances R_1 and R_2 are included to account for the losses (power dissipation) in the coils. The transformer is said to be *linear* if the coils are wound on a magnetically linear

A linear transformer may also be regarded as one whose flux is proportional to the currents in its windings.

material—a material for which the magnetic permeability is constant. Such materials include air, plastic, Bakelite, and wood. In fact, most materials are magnetically linear. Linear transformers are sometimes called *air-core transformers*, although not all of them are necessarily air-core. They are used in radio and TV sets. Figure 13.20 portrays different types of transformers.

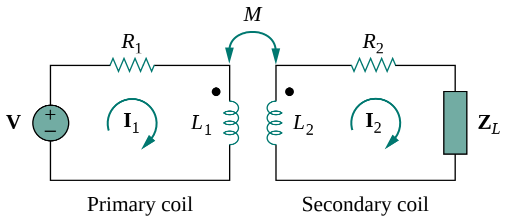
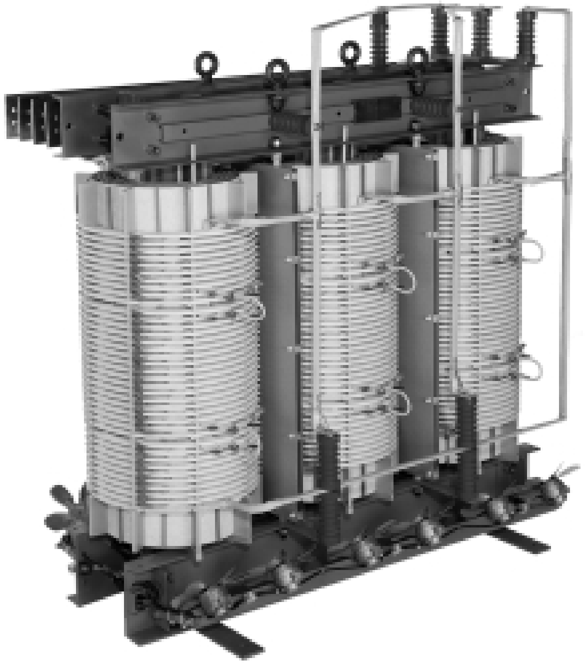


Figure 13.19 A linear transformer.



(a)

(b)

Figure 13.20 Different types of transformers: (a) copper wound dry power transformer, (b) audio transformers. *(Courtesy of: (a) Electric Service Co., (b) Jensen Transformers.)*

We would like to obtain the input impedance $\mathbf{Z}_{in}$ as seen from the source, because $\mathbf{Z}_{in}$ governs the behavior of the primary circuit. Applying KVL to the two meshes in Fig. 13.19 gives

$$\mathbf{V} = (R_1 + j\omega L_1)\mathbf{I}_1 - j\omega M\mathbf{I}_2 \tag{13.40a}$$

$$0 = -j\omega M\mathbf{I}_1 + (R_2 + j\omega L_2 + \mathbf{Z}_L)\mathbf{I}_2 \tag{13.40b}$$

In Eq. (13.40b), we express $\mathbf{I}_2$ in terms of $\mathbf{I}_1$ and substitute it into Eq. (13.40a). We get the input impedance as

$$\mathbf{Z}_{\text{in}} = \frac{\mathbf{V}}{\mathbf{I}_1} = R_1 + j\omega L_1 + \frac{\omega^2 M^2}{R_2 + j\omega L_2 + \mathbf{Z}_L} \tag{13.41}$$

Notice that the input impedance comprises two terms. The first term, $(R_1 + j\omega L_1)$, is the primary impedance. The second term is due to the coupling between the primary and secondary windings. It is as though this impedance is reflected to the primary. Thus, it is known as the *reflected impedance* $\mathbf{Z}_R$, and

Some authors call this the *coupled impedance*.

$$\mathbf{Z}_R = \frac{\omega^2 M^2}{R_2 + j\omega L_2 + \mathbf{Z}_L} \tag{13.42}$$

It should be noted that the result in Eq. (13.41) or (13.42) is not affected by the location of the dots on the transformer, because the same result is produced when M is replaced by $-M$.

The little bit of experience gained in Sections 13.2 and 13.3 in analyzing magnetically coupled circuits is enough to convince anyone that analyzing these circuits is not as easy as circuits in previous chapters. For this reason, it is sometimes convenient to replace a magnetically coupled circuit by an equivalent circuit with no magnetic coupling. We want to replace the linear transformer in Fig. 13.19 by an equivalent T or Π circuit, a circuit that would have no mutual inductance. Ignore the resistances of the coils and assume that the coils have a common ground as shown in Fig. 13.21. The assumption of a common ground for the two coils is a major restriction of the equivalent circuits. A common ground is imposed on the linear transformer in Fig. 13.21 in view of the necessity of having a common ground in the equivalent T or Π circuit; see Figs. 13.22 and 13.23.

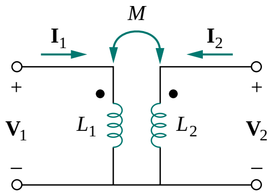


Figure 13.21 Determining the equivalent circuit of a linear transformer.

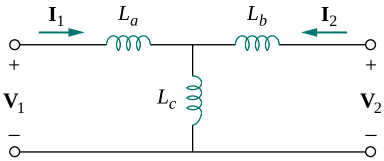


Figure 13.22 An equivalent T circuit.

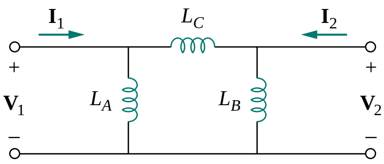


Figure 13.23 An equivalent Π circuit.

The voltage-current relationships for the primary and secondary coils give the matrix equation

$$\begin{bmatrix} \mathbf{V}_1 \\ \mathbf{V}_2 \end{bmatrix} = \begin{bmatrix} j\omega L_1 & j\omega M \\ j\omega M & j\omega L_2 \end{bmatrix} \begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \end{bmatrix} \tag{13.43}$$

By matrix inversion, this can be written as

$$\begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \end{bmatrix} = \begin{bmatrix} \dfrac{L_2}{j\omega(L_1L_2 - M^2)} & \dfrac{-M}{j\omega(L_1L_2 - M^2)} \\ \dfrac{-M}{j\omega(L_1L_2 - M^2)} & \dfrac{L_1}{j\omega(L_1L_2 - M^2)} \end{bmatrix} \begin{bmatrix} \mathbf{V}_1 \\ \mathbf{V}_2 \end{bmatrix} \tag{13.44}$$

Our goal is to match Eqs. (13.43) and (13.44) with the corresponding equations for the T and Π networks.

For the T (or Y) network of Fig. 13.22, mesh analysis provides the terminal equations as

$$\begin{bmatrix} \mathbf{V}_1 \\ \mathbf{V}_2 \end{bmatrix} = \begin{bmatrix} j\omega(L_a + L_c) & j\omega L_c \\ j\omega L_c & j\omega(L_b + L_c) \end{bmatrix} \begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \end{bmatrix} \tag{13.45}$$

If the circuits in Figs. 13.21 and 13.22 are equivalents, Eqs. (13.43) and (13.45) must be identical. Equating terms in the impedance matrices of

Eqs. (13.43) and (13.45) leads to

$$L_a = L_1 - M, \qquad L_b = L_2 - M, \qquad L_c = M \tag{13.46}$$

For the Π (or Δ) network in Fig. 13.23, nodal analysis gives the terminal equations as

$$\begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \end{bmatrix} = \begin{bmatrix} \dfrac{1}{j\omega L_A} + \dfrac{1}{j\omega L_C} & -\dfrac{1}{j\omega L_C} \\ -\dfrac{1}{j\omega L_C} & \dfrac{1}{j\omega L_B} + \dfrac{1}{j\omega L_C} \end{bmatrix} \begin{bmatrix} \mathbf{V}_1 \\ \mathbf{V}_2 \end{bmatrix} \tag{13.47}$$

Equating terms in admittance matrices of Eqs. (13.44) and (13.47), we obtain

$$L_A = \frac{L_1L_2 - M^2}{L_2 - M}, \qquad L_B = \frac{L_1L_2 - M^2}{L_1 - M}$$
$$L_C = \frac{L_1L_2 - M^2}{M} \tag{13.48}$$

Note that in Figs. 13.23 and 13.24, the inductors are not magnetically coupled. Also note that changing the locations of the dots in Fig. 13.21 can cause M to become $-M$. As Example 13.6 illustrates, a negative value of M is physically unrealizable but the equivalent model is still mathematically valid.

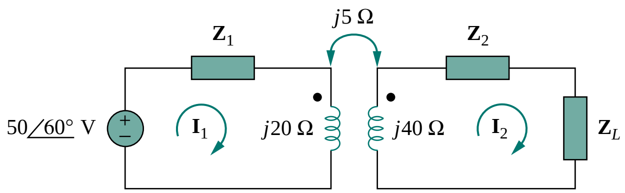


Figure 13.24 For Example 13.4.

EXAMPLE 13.4

In the circuit of Fig. 13.24, calculate the input impedance and current $\mathbf{I}_1$. Take $\mathbf{Z}_1 = 60 - j100\ \Omega$, $\mathbf{Z}_2 = 30 + j40\ \Omega$, and $\mathbf{Z}_L = 80 + j60\ \Omega$.

Solution:

From Eq. (13.41),

$$\begin{aligned} \mathbf{Z}_{\text{in}} &= \mathbf{Z}_1 + j20 + \frac{(5)^2}{j40 + \mathbf{Z}_2 + \mathbf{Z}_L} \\ &= 60 - j100 + j20 + \frac{25}{110 + j140} \\ &= 60 - j80 + 0.14\underline{/-51.84^\circ} \\ &= 60.09 - j80.11 = 100.14\underline{/-53.1^\circ}\ \Omega \end{aligned}$$

Thus,

$$\mathbf{I}_1 = \frac{\mathbf{V}}{\mathbf{Z}_{\text{in}}} = \frac{50\underline{/60^\circ}}{100.14\underline{/-53.1^\circ}} = 0.5\underline{/113.1^\circ} \text{ A}$$

PRACTICE PROBLEM 13.4

Find the input impedance of the circuit of Fig. 13.25 and the current from the voltage source.

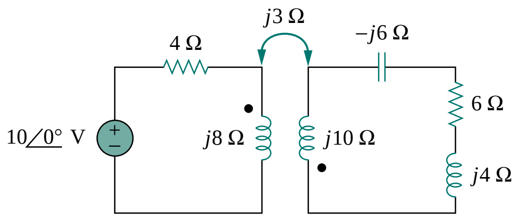


Figure 13.25 For Practice Prob. 13.4.

Answer: $8.58\underline{/58.05^\circ}\ \Omega$, $1.165\underline{/-58.05^\circ}$ A.

EXAMPLE 13.5

Determine the T-equivalent circuit of the linear transformer in Fig. 13.26(a).

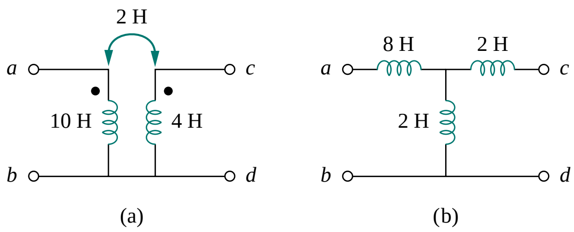


Figure 13.26 For Example 13.5: (a) a linear transformer, (b) its T-equivalent circuit.

Solution:

Given that $L_1 = 10$, $L_2 = 4$, and $M = 2$, the T equivalent network has the following parameters:

$$L_a = L_1 - M = 10 - 2 = 8 \text{ H}$$

$$L_b = L_2 - M = 4 - 2 = 2 \text{ H}, \qquad L_c = M = 2 \text{ H}$$

The T-equivalent circuit is shown in Fig. 13.26(b). We have assumed that reference directions for currents and voltage polarities in the primary and secondary windings conform to those in Fig. 13.21. Otherwise, we may need to replace M with $-M$, as Example 13.6 illustrates.

PRACTICE PROBLEM 13.5

For the linear transformer in Fig. 13.26 (a), find the Π equivalent network.

Answer: $L_A = 18$ H, $L_B = 4.5$ H, $L_C = 18$ H.

EXAMPLE 13.6

Solve for $\mathbf{I}_1$, $\mathbf{I}_2$, and $\mathbf{V}_o$ in Fig. 13.27 (the same circuit as for Practice Prob. 13.1) using the T-equivalent circuit for the linear transformer.

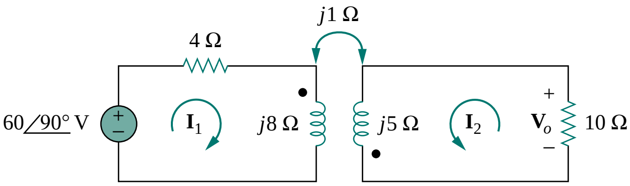


Figure 13.27 For Example 13.6.

Solution:

Notice that the circuit in Fig. 13.27 is the same as that in Fig. 13.10 except that the reference direction for current $\mathbf{I}_2$ has been reversed, just to make the reference directions for the currents for the magnetically coupled coils conform with those in Fig. 13.21.

We need to replace the magnetically coupled coils with the T-equivalent circuit. The relevant portion of the circuit in Fig. 13.27 is shown in Fig. 13.28(a). Comparing Fig. 13.28(a) with Fig. 13.21 shows that there are two differences. First, due to the current reference directions and voltage polarities, we need to replace M by $-M$ to make Fig. 13.28(a) conform with Fig. 13.21. Second, the circuit in Fig. 13.21 is in the time-domain, whereas the circuit in Fig. 13.28(a) is in the frequency-domain. The difference is the factor $j\omega$; that is, L in Fig. 13.21 has been replaced with $j\omega L$ and M with $j\omega M$. Since ω is not specified, we can assume $\omega = 1$ or any other value; it really does not matter. With these two differences in mind,

$$L_a = L_1 - (-M) = 8 + 1 = 9 \text{ H}$$

$$L_b = L_2 - (-M) = 5 + 1 = 6 \text{ H}, \qquad L_c = -M = -1 \text{ H}$$

Thus, the T-equivalent circuit for the coupled coils is as shown in Fig. 13.28(b).

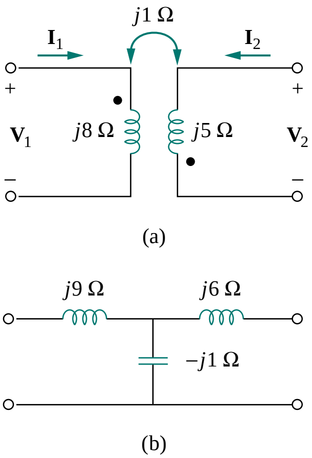


Figure 13.28 For Example 13.6: (a) circuit for coupled coils of Fig. 13.27, (b) T-equivalent circuit.

Inserting the T-equivalent circuit in Fig. 13.28(b) to replace the two coils in Fig. 13.27 gives the equivalent circuit in Fig. 13.29, which can be solved using nodal or mesh analysis. Applying mesh analysis, we obtain

$$j6 = \mathbf{I}_1(4 + j9 - j1) + \mathbf{I}_2(-j1) \tag{13.6.1}$$

and

$$0 = \mathbf{I}_1(-j1) + \mathbf{I}_2(10 + j6 - j1) \tag{13.6.2}$$

From Eq. (13.6.2),

$$\mathbf{I}_1 = \frac{(10 + j5)}{j}\mathbf{I}_2 = (5 - j10)\mathbf{I}_2 \tag{13.6.3}$$

Substituting Eq. (13.6.3) into Eq. (13.6.1) gives

$$j6 = (4 + j8)(5 - j10)\mathbf{I}_2 - j\mathbf{I}_2 = (100 - j)\mathbf{I}_2 \simeq 100\mathbf{I}_2$$

Since 100 is very large compared to 1, the imaginary part of $(100 - j)$ can be ignored so that $100 - j \simeq 100$. Hence,

$$\mathbf{I}_2 = \frac{j6}{100} = j0.06 = 0.06\underline{/90^\circ} \text{ A}$$

From Eq. (13.6.3),

$$\mathbf{I}_1 = (5 - j10)j0.06 = 0.6 + j0.3 \text{ A}$$

and

$$\mathbf{V}_o = -10\mathbf{I}_2 = -j0.6 = 0.6\underline{/-90^\circ} \text{ V}$$

This agrees with the answer to Practice Prob. 13.1. Of course, the direction of $\mathbf{I}_2$ in Fig. 13.10 is opposite to that in Fig. 13.27. This will not affect $\mathbf{V}_o$, but the value of $\mathbf{I}_2$ in this example is the negative of that of $\mathbf{I}_2$ in Practice Prob. 13.1. The advantage of using the T-equivalent model for the magnetically coupled coils is that in Fig. 13.29 we do not need to bother with the dot on the coupled coils.

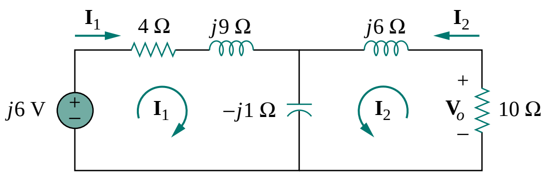


Figure 13.29 For Example 13.6.

PRACTICE PROBLEM 13.6

Solve the problem in Example 13.1 (see Fig. 13.9) using the T-equivalent model for the magnetically coupled coils.

Answer: $13\underline{/-49.4^\circ}$ A, $2.91\underline{/14.04^\circ}$ A.

13.5 IDEAL TRANSFORMERS

An ideal transformer is one with perfect coupling ($k = 1$). It consists of two (or more) coils with a large number of turns wound on a common core of high permeability. Because of this high permeability of the core, the flux links all the turns of both coils, thereby resulting in a perfect coupling.

To see how an ideal transformer is the limiting case of two coupled inductors where the inductances approach infinity and the coupling is perfect, let us reexamine the circuit in Fig. 13.14. In the frequency domain,

$$\mathbf{V}_1 = j\omega L_1\mathbf{I}_1 + j\omega M\mathbf{I}_2 \tag{13.49a}$$

$$\mathbf{V}_2 = j\omega M\mathbf{I}_1 + j\omega L_2\mathbf{I}_2 \tag{13.49b}$$

From Eq. (13.49a), $\mathbf{I}_1 = (\mathbf{V}_1 - j\omega M\mathbf{I}_2)/j\omega L_1$. Substituting this in Eq. (13.49b) gives

$$\mathbf{V}_2 = j\omega L_2\mathbf{I}_2 + \frac{M\mathbf{V}_1}{L_1} - \frac{j\omega M^2\mathbf{I}_2}{L_1}$$

But $M = \sqrt{L_1 L_2}$ for perfect coupling ($k = 1$). Hence,

$$\mathbf{V}_2 = j\omega L_2\mathbf{I}_2 + \frac{\sqrt{L_1L_2}\mathbf{V}_1}{L_1} - \frac{j\omega L_1L_2\mathbf{I}_2}{L_1} = \sqrt{\frac{L_2}{L_1}}\mathbf{V}_1 = n\mathbf{V}_1$$

where $n = \sqrt{L_2/L_1}$ and is called the *turns ratio*. As $L_1, L_2, M \rightarrow \infty$ such that n remains the same, the coupled coils become an ideal transformer. A transformer is said to be ideal if it has the following properties:

1. Coils have very large reactances ($L_1, L_2, M \rightarrow \infty$).
2. Coupling coefficient is equal to unity ($k = 1$).
3. Primary and secondary coils are lossless ($R_1 = 0 = R_2$).

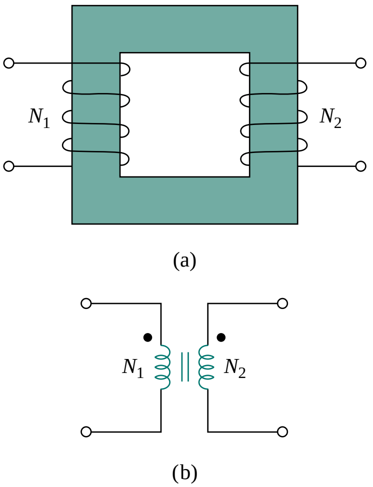


Figure 13.30 (a) Ideal transformer, (b) circuit symbol for ideal transformers.

> An ideal transformer is a unity-coupled, lossless transformer in which the primary and secondary coils have infinite self-inductances.

Iron-core transformers are close approximations to ideal transformers. These are used in power systems and electronics.

Figure 13.30(a) shows a typical ideal transformer; the circuit symbol is in Fig. 13.30(b). The vertical lines between the coils indicate an iron core as distinct from the air core used in linear transformers. The primary winding has N_1 turns; the secondary winding has N_2 turns.

When a sinusoidal voltage is applied to the primary winding as shown in Fig. 13.31, the same magnetic flux ϕ goes through both windings. According to Faraday's law, the voltage across the primary winding is

$$v_1 = N_1 \frac{d\phi}{dt} \tag{13.50a}$$

while that across the secondary winding is

$$v_2 = N_2 \frac{d\phi}{dt} \tag{13.50b}$$

Dividing Eq. (13.50b) by Eq. (13.50a), we get

$$\frac{v_2}{v_1} = \frac{N_2}{N_1} = n \tag{13.51}$$

where n is, again, the *turns ratio* or *transformation ratio*. We can use the phasor voltages $\mathbf{V}_1$ and $\mathbf{V}_2$ rather than the instantaneous values v_1 and v_2. Thus, Eq. (13.51) may be written as

$$\boxed{\frac{\mathbf{V}_2}{\mathbf{V}_1} = \frac{N_2}{N_1} = n} \tag{13.52}$$

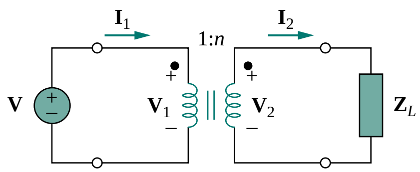


Figure 13.31 Relating primary and secondary quantities in an ideal transformer.

For the reason of power conservation, the energy supplied to the primary must equal the energy absorbed by the secondary, since there are no losses in an ideal transformer. This implies that

$$v_1 i_1 = v_2 i_2 \tag{13.53}$$

In phasor form, Eq. (13.53) in conjunction with Eq. (13.52) becomes

$$\frac{\mathbf{I}_1}{\mathbf{I}_2} = \frac{\mathbf{V}_2}{\mathbf{V}_1} = n \tag{13.54}$$

showing that the primary and secondary currents are related to the turns ratio in the inverse manner as the voltages. Thus,

$$\boxed{\frac{\mathbf{I}_2}{\mathbf{I}_1} = \frac{N_1}{N_2} = \frac{1}{n}} \tag{13.55}$$

When $n = 1$, we generally call the transformer an *isolation transformer*. The reason will become obvious in Section 13.9.1. If $n > 1$, we have a *step-up transformer*, as the voltage is increased from primary to secondary ($\mathbf{V}_2 > \mathbf{V}_1$). On the other hand, if $n < 1$, the transformer is a *step-down transformer*, since the voltage is decreased from primary to secondary ($\mathbf{V}_2 < \mathbf{V}_1$).

> A **step-down transformer** is one whose secondary voltage is less than its primary voltage.

> A **step-up transformer** is one whose secondary voltage is greater than its primary voltage.

The ratings of transformers are usually specified as V_1/V_2. A transformer with rating 2400/120 V should have 2400 V on the primary and 120 in the secondary (i.e., a step-down transformer). Keep in mind that the voltage ratings are in rms.

Power companies often generate at some convenient voltage and use a step-up transformer to increase the voltage so that the power can be transmitted at very high voltage and low current over transmission lines, resulting in significant cost savings. Near residential consumer premises, step-down transformers are used to bring the voltage down to 120 V. Section 13.9.3 will elaborate on this.

It is important that we know how to get the proper polarity of the voltages and the direction of the currents for the transformer in Fig. 13.31. If the polarity of $\mathbf{V}_1$ or $\mathbf{V}_2$ or the direction of $\mathbf{I}_1$ or $\mathbf{I}_2$ is changed, n in Eqs. (13.51) to (13.55) may need to be replaced by $-n$. The two simple rules to follow are:

1. If $\mathbf{V}_1$ and $\mathbf{V}_2$ are *both* positive or both negative at the dotted terminals, use $+n$ in Eq. (13.52). Otherwise, use $-n$.
2. If $\mathbf{I}_1$ and $\mathbf{I}_2$ *both* enter into or both leave the dotted terminals, use $-n$ in Eq. (13.55). Otherwise, use $+n$.

The rules are demonstrated with the four circuits in Fig. 13.32.

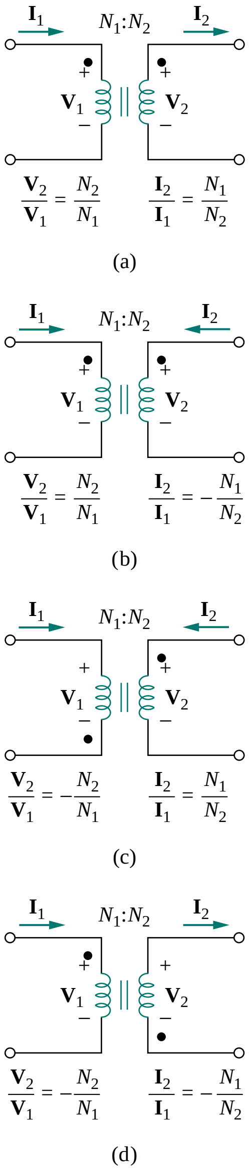


Figure 13.32 Typical circuits illustrating proper voltage polarities and current directions in an ideal transformer.

Using Eqs. (13.52) and (13.55), we can always express $\mathbf{V}_1$ in terms of $\mathbf{V}_2$ and $\mathbf{I}_1$ in terms of $\mathbf{I}_2$, or vice versa:

$$\mathbf{V}_1 = \frac{\mathbf{V}_2}{n} \qquad \text{or} \qquad \mathbf{V}_2 = n\mathbf{V}_1 \tag{13.56}$$

$$\mathbf{I}_1 = n\mathbf{I}_2 \qquad \text{or} \qquad \mathbf{I}_2 = \frac{\mathbf{I}_1}{n} \tag{13.57}$$

The complex power in the primary winding is

$$\mathbf{S}_1 = \mathbf{V}_1\mathbf{I}_1^* = \frac{\mathbf{V}_2}{n}(n\mathbf{I}_2)^* = \mathbf{V}_2\mathbf{I}_2^* = \mathbf{S}_2 \tag{13.58}$$

showing that the complex power supplied to the primary is delivered to the secondary without loss. The transformer absorbs no power. Of course, we should expect this, since the ideal transformer is lossless. The input impedance as seen by the source in Fig. 13.31 is found from Eqs. (13.56) and (13.57) as

$$\mathbf{Z}_{\text{in}} = \frac{\mathbf{V}_1}{\mathbf{I}_1} = \frac{1}{n^2}\frac{\mathbf{V}_2}{\mathbf{I}_2} \tag{13.59}$$

It is evident from Fig. 13.31 that $\mathbf{V}_2/\mathbf{I}_2 = \mathbf{Z}_L$, so that

$$\mathbf{Z}_{\text{in}} = \frac{\mathbf{Z}_L}{n^2} \tag{13.60}$$

Notice that an ideal transformer reflects an impedance as the square of the turns ratio.

The input impedance is also called the *reflected impedance*, since it appears as if the load impedance is reflected to the primary side. This ability of the transformer to transform a given impedance into another impedance provides us a means of *impedance matching* to ensure maximum power transfer. The idea of impedance matching is very useful in practice and will be discussed more in Section 13.9.2.

In analyzing a circuit containing an ideal transformer, it is common practice to eliminate the transformer by reflecting impedances and sources from one side of the transformer to the other. In the circuit of Fig. 13.33, suppose we want to reflect the secondary side of the circuit to the primary side. We find the Thevenin equivalent of the circuit to the right of the terminals a-b. We obtain $\mathbf{V}_{\text{Th}}$ as the open-circuit voltage at terminals a-b, as shown in Fig. 13.34(a). Since terminals a-b are open, $\mathbf{I}_1 = 0 = \mathbf{I}_2$ so that $\mathbf{V}_2 = \mathbf{V}_{s2}$. Hence, from Eq. (13.56),

$$\mathbf{V}_{\text{Th}} = \mathbf{V}_1 = \frac{\mathbf{V}_2}{n} = \frac{\mathbf{V}_{s2}}{n} \tag{13.61}$$

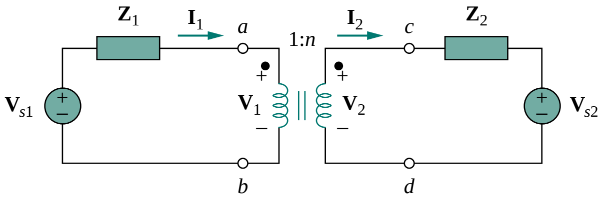


Figure 13.33 Ideal transformer circuit whose equivalent circuits are to be found.

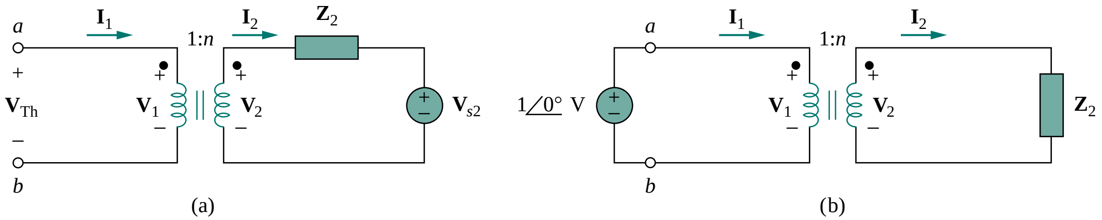


Figure 13.34 (a) Obtaining $\mathbf{V}_{Th}$ for the circuit in Fig. 13.33, (b) obtaining $\mathbf{Z}_{Th}$ for the circuit in Fig. 13.33.

To get $\mathbf{Z}_{Th}$, we remove the voltage source in the secondary winding and insert a unit source at terminals a-b, as in Fig. 13.34(b). From Eqs. (13.56) and (13.57), $\mathbf{I}_1 = n\mathbf{I}_2$ and $\mathbf{V}_1 = \mathbf{V}_2/n$, so that

$$\mathbf{Z}_{Th} = \frac{\mathbf{V}_1}{\mathbf{I}_1} = \frac{\mathbf{V}_2/n}{n\mathbf{I}_2} = \frac{\mathbf{Z}_2}{n^2}, \qquad \mathbf{V}_2 = \mathbf{Z}_2\mathbf{I}_2 \tag{13.62}$$

which is what we should have expected from Eq. (13.60). Once we have $\mathbf{V}_{Th}$ and $\mathbf{Z}_{Th}$, we add the Thevenin equivalent to the part of the circuit in Fig. 13.33 to the left of terminals a-b. Figure 13.35 shows the result.

The general rule for eliminating the transformer and reflecting the secondary circuit to the primary side is: divide the secondary impedance by n^2, divide the secondary voltage by n, and multiply the secondary current by n.

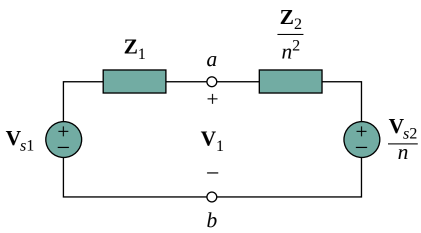


Figure 13.35 Equivalent circuit for Fig. 13.33 obtained by reflecting the secondary circuit to the primary side.

We can also reflect the primary side of the circuit in Fig. 13.33 to the secondary side. Figure 13.36 shows the equivalent circuit.

The rule for eliminating the transformer and reflecting the primary circuit to the secondary side is: multiply the primary impedance by n^2, multiply the primary voltage by n, and divide the primary current by n.

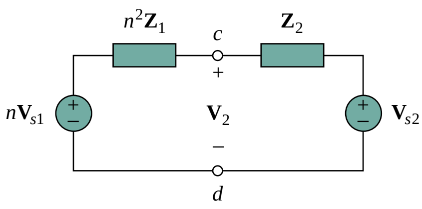


Figure 13.36 Equivalent circuit for Fig. 13.33 obtained by reflecting the primary circuit to the secondary side.

According to Eq. (13.58), the power remains the same, whether calculated on the primary or the secondary side. But realize that this reflection approach only applies if there are no external connections between the primary and secondary windings. When we have external connections between the primary and secondary windings, we simply use regular mesh and nodal analysis. Examples of circuits where there are external connections between the primary and secondary windings are in Figs. 13.39 and 13.40. Also note that if the locations of the dots in Fig. 13.33 are changed, we might have to replace n by $-n$ in order to obey the dot rule, illustrated in Fig. 13.32.

EXAMPLE 13.7

An ideal transformer is rated at 2400/120 V, 9.6 kVA, and has 50 turns on the secondary side. Calculate: (a) the turns ratio, (b) the number of turns on the primary side, and (c) the current ratings for the primary and secondary windings.

Solution:

(a) This is a step-down transformer, since $V_1 = 2400$ V $> V_2 = 120$ V.

$$n = \frac{V_2}{V_1} = \frac{120}{2400} = 0.05$$

(b)

$$n = \frac{N_2}{N_1} \qquad \Longrightarrow \qquad 0.05 = \frac{50}{N_1}$$

or

$$N_1 = \frac{50}{0.05} = 1000 \text{ turns}$$

(c) $S = V_1 I_1 = V_2 I_2 = 9.6$ kVA. Hence,

$$I_1 = \frac{9600}{V_1} = \frac{9600}{2400} = 4 \text{ A}$$

$$I_2 = \frac{9600}{V_2} = \frac{9600}{120} = 80 \text{ A} \qquad \text{or} \qquad I_2 = \frac{I_1}{n} = \frac{4}{0.05} = 80 \text{ A}$$

PRACTICE PROBLEM 13.7

The primary current to an ideal transformer rated at 3300/110 V is 3 A. Calculate: (a) the turns ratio, (b) the kVA rating, (c) the secondary current.

Answer: (a) 1/30, (b) 9.9 kVA, (c) 90 A.

EXAMPLE 13.8

For the ideal transformer circuit of Fig. 13.37, find: (a) the source current $\mathbf{I}_1$, (b) the output voltage $\mathbf{V}_o$, and (c) the complex power supplied by the source.

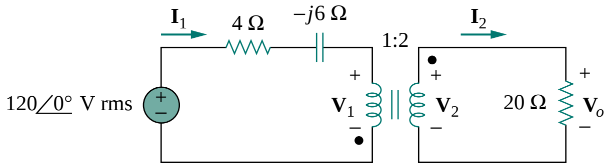


Figure 13.37 For Example 13.8.

Solution:

(a) The 20-Ω impedance can be reflected to the primary side and we get

$$\mathbf{Z}_R = \frac{20}{n^2} = \frac{20}{4} = 5 \ \Omega$$

Thus,

$$\mathbf{Z}_{\text{in}} = 4 - j6 + \mathbf{Z}_R = 9 - j6 = 10.82\underline{/-33.69^\circ} \ \Omega$$

$$\mathbf{I}_1 = \frac{120\underline{/0^\circ}}{\mathbf{Z}_{\text{in}}} = \frac{120\underline{/0^\circ}}{10.82\underline{/-33.69^\circ}} = 11.09\underline{/33.69^\circ} \text{ A}$$

(b) Since both $\mathbf{I}_1$ and $\mathbf{I}_2$ leave the dotted terminals,

$$\mathbf{I}_2 = -\frac{1}{n}\mathbf{I}_1 = -5.545\underline{/33.69^\circ}\text{ A}$$

$$\mathbf{V}_o = 20\mathbf{I}_2 = 110.9\underline{/213.69^\circ}\text{ V}$$

(c) The complex power supplied is

$$\mathbf{S} = \mathbf{V}_s\mathbf{I}_1^* = (120\underline{/0^\circ})(11.09\underline{/-33.69^\circ}) = 1330.8\underline{/-33.69^\circ}\text{ VA}$$

PRACTICE PROBLEM 13.8

In the ideal transformer circuit of Fig. 13.38, find $\mathbf{V}_o$ and the complex power supplied by the source.

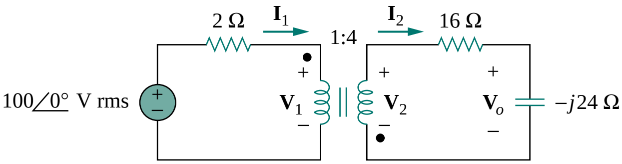


Figure 13.38 For Practice Prob. 13.8.

Answer: $178.9\underline{/116.56^\circ}$ V, $2981.5\underline{/-26.56^\circ}$ VA.

EXAMPLE 13.9

Calculate the power supplied to the 10-Ω resistor in the ideal transformer circuit of Fig. 13.39.

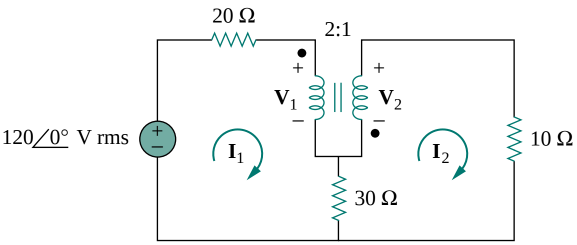


Figure 13.39 For Example 13.9.

Solution:

Reflection to the secondary or primary side cannot be done with this circuit: there is direct connection between the primary and secondary sides due to the 30-Ω resistor. We apply mesh analysis. For mesh 1,

$$-120 + (20 + 30)\mathbf{I}_1 - 30\mathbf{I}_2 + \mathbf{V}_1 = 0$$

or

$$50\mathbf{I}_1 - 30\mathbf{I}_2 + \mathbf{V}_1 = 120 \qquad (13.9.1)$$

For mesh 2,

$$-\mathbf{V}_2 + (10 + 30)\mathbf{I}_2 - 30\mathbf{I}_1 = 0$$

or

$$-30\mathbf{I}_1 + 40\mathbf{I}_2 - \mathbf{V}_2 = 0 \tag{13.9.2}$$

At the transformer terminals,

$$\mathbf{V}_2 = -\frac{1}{2}\mathbf{V}_1 \tag{13.9.3}$$

$$\mathbf{I}_2 = -2\mathbf{I}_1 \tag{13.9.4}$$

(Note that $n = 1/2$.) We now have four equations and four unknowns, but our goal is to get $\mathbf{I}_2$. So we substitute for $\mathbf{V}_1$ and $\mathbf{I}_1$ in terms of $\mathbf{V}_2$ and $\mathbf{I}_2$ in Eqs. (13.9.1) and (13.9.2). Equation (13.9.1) becomes

$$-55\mathbf{I}_2 - 2\mathbf{V}_2 = 120 \tag{13.9.5}$$

and Eq. (13.9.2) becomes

$$15\mathbf{I}_2 + 40\mathbf{I}_2 - \mathbf{V}_2 = 0 \qquad \Longrightarrow \qquad \mathbf{V}_2 = 55\mathbf{I}_2 \tag{13.9.6}$$

Substituting Eq. (13.9.6) in Eq. (13.9.5),

$$-165\mathbf{I}_2 = 120 \qquad \Longrightarrow \qquad \mathbf{I}_2 = -\frac{120}{165} = -0.7272 \text{ A}$$

The power absorbed by the 10-Ω resistor is

$$P = (-0.7272)^2(10) = 5.3 \text{ W}$$

PRACTICE PROBLEM 13.9

Find $\mathbf{V}_o$ in the circuit in Fig. 13.40.

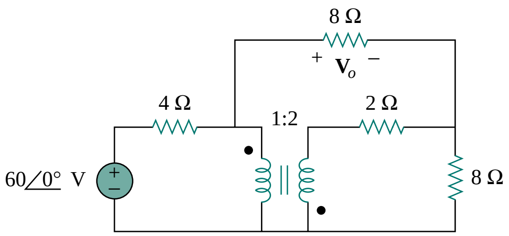


Figure 13.40 For Practice Prob. 13.9.

Answer: 24 V.

13.6 IDEAL AUTOTRANSFORMERS

Unlike the conventional two-winding transformer we have considered so far, an *autotransformer* has a single continuous winding with a connection point called a *tap* between the primary and secondary sides. The tap is often adjustable so as to provide the desired turns ratio for stepping up or stepping down the voltage. This way, a variable voltage is provided to the load connected to the autotransformer.

> An **autotransformer** is a transformer in which both the primary and the secondary are in a single winding.

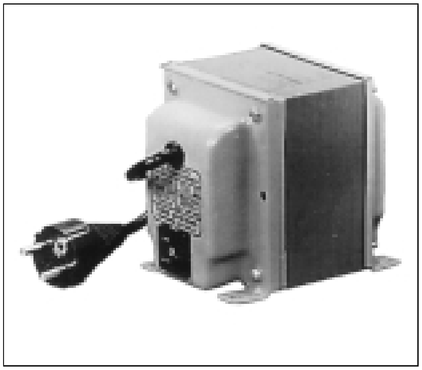


Figure 13.41 A typical autotransformer. *(Courtesy of Todd Systems, Inc.)*

Figure 13.41 shows a typical autotransformer. As shown in Fig. 13.42, the autotransformer can operate in the step-down or step-up mode. The autotransformer is a type of power transformer. Its major advantage over the two-winding transformer is its ability to transfer larger apparent power. Example 13.10 will demonstrate this. Another advantage is that an autotransformer is smaller and lighter than an equivalent two-winding transformer. However, since both the primary and secondary windings are one winding, *electrical isolation* (no direct electrical connection) is lost. (We will see how the property of electrical isolation in the conventional transformer is practically employed in Section 13.9.1.) The lack of electrical isolation between the primary and secondary windings is a major disadvantage of the autotransformer.

Some of the formulas we derived for ideal transformers apply to ideal autotransformers as well. For the step-down autotransformer circuit of Fig. 13.42(a), Eq. (13.52) gives

$$\boxed{\frac{\mathbf{V}_1}{\mathbf{V}_2} = \frac{N_1 + N_2}{N_2} = 1 + \frac{N_1}{N_2}} \tag{13.63}$$

As an ideal autotransformer, there are no losses, so the complex power remains the same in the primary and secondary windings:

$$\mathbf{S}_1 = \mathbf{V}_1\mathbf{I}_1^* = \mathbf{S}_2 = \mathbf{V}_2\mathbf{I}_2^* \tag{13.64}$$

Equation (13.64) can also be expressed with rms values as

$$V_1 I_1 = V_2 I_2$$

or

$$\frac{V_2}{V_1} = \frac{I_1}{I_2} \tag{13.65}$$

Thus, the current relationship is

$$\frac{\mathbf{I}_1}{\mathbf{I}_2} = \frac{N_2}{N_1 + N_2} \tag{13.66}$$

For the step-up autotransformer circuit of Fig. 13.42(b),

$$\frac{\mathbf{V}_1}{N_1} = \frac{\mathbf{V}_2}{N_1 + N_2}$$

or

$$\boxed{\frac{\mathbf{V}_1}{\mathbf{V}_2} = \frac{N_1}{N_1 + N_2}} \tag{13.67}$$

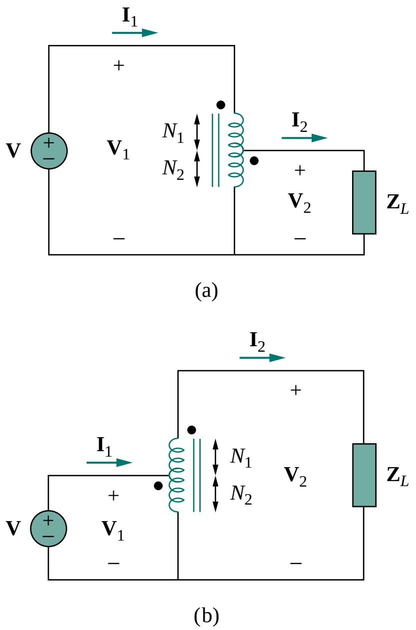


Figure 13.42 (a) Step-down autotransformer, (b) step-up autotransformer.

The complex power given by Eq. (13.64) also applies to the step-up autotransformer so that Eq. (13.65) again applies. Hence, the current relationship is

$$\frac{\mathbf{I}_1}{\mathbf{I}_2} = \frac{N_1 + N_2}{N_1} = 1 + \frac{N_2}{N_1} \tag{13.68}$$

A major difference between conventional transformers and autotransformers is that the primary and secondary sides of the autotransformer are not only coupled magnetically but also coupled conductively. The autotransformer can be used in place of a conventional transformer when electrical isolation is not required.

EXAMPLE 13.10

Compare the power ratings of the two-winding transformer in Fig. 13.43(a) and the autotransformer in Fig. 13.43(b).

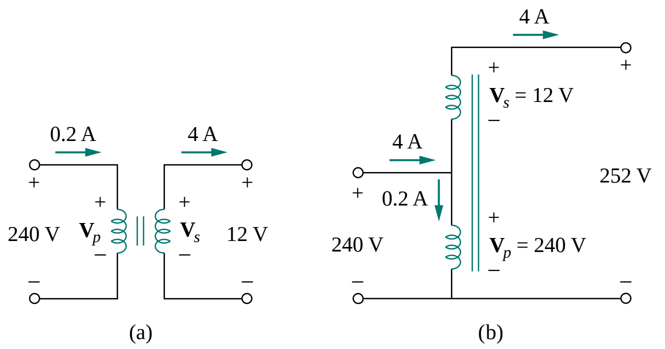


Figure 13.43 For Example 13.10.

Solution:

Although the primary and secondary windings of the autotransformer are together as a continuous winding, they are separated in Fig. 13.43(b) for clarity. We note that the current and voltage of each winding of the autotransformer in Fig. 13.43(b) are the same as those for the two-winding transformer in Fig. 13.43(a). This is the basis of comparing their power ratings.

For the two-winding transformer, the power rating is

$$S_1 = 0.2(240) = 48 \text{ VA} \quad \text{or} \quad S_2 = 4(12) = 48 \text{ VA}$$

For the autotransformer, the power rating is

$$S_1 = 4.2(240) = 1008 \text{ VA} \quad \text{or} \quad S_2 = 4(252) = 1008 \text{ VA}$$

which is 21 times the power rating of the two-winding transformer.

PRACTICE PROBLEM 13.10

Refer to Fig. 13.43. If the two-winding transformer is a 60-VA, 120 V/10 V transformer, what is the power rating of the autotransformer?

Answer: 780 VA.